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Darboux's problem of quadratic integrals

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Abstract. The problem of determining all standard classical Hamiltonians in two dimensions with Euclidean metric which admit constants of motion quadratic in the momenta is resolved. Several general results are given which make it obvious that the systems found do possess such integrals.

1. Introduction

In this note I wish to resolve a problem which was first considered, it seems, by Darboux (1901). Quite simply the problem may be stated as follows: suppose that H is a Hamiltonian function for a system of two degrees of freedom and that H has the form given by

$$H=\tfrac{1}{2}(p_x^2+p_y^2)+V(x,y);$$

then determine all V for which there is a constant of motion quadratic in the momenta besides H itself. Darboux examined the problem and gave a partial solution but in the classical vein ignored certain exceptional cases. A similar discussion may be found in Whittaker (1937).

Before embarking on the problem above, I want to state several general results which allow certain constants of motion to be written down directly by examining the form of the Hamiltonian. These are, I think, interesting in their own right but have the added advantage that two of them will corroborate our computations; they embrace all possible solutions to the problem.

2. Results guaranteeing existence of certain integrals

For this section we shall suppose that we are working on a $2m$-dimensional space with coordinates (x^i, p_j). Here and throughout I employ the notation of classical tensor calculus with δ_{ij}, summation convention etc. Also $\{,\}$ denotes the Poisson bracket of functions.

Proposition 2.1. Suppose H and f are functions of (x^i, p_j) and that $\{H,f\}=0$ and $\{p_i\partial H/\partial p_i, f\}=0$; then $\{H, p_i\partial f/\partial p_i\}=0$.

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Although this may look very coordinate dependent it is in fact intrinsic because the vector field $p_i\partial/\partial p_i$ is the canonical radial vector field. It is easily proved from the formula of differential geometry

$$\mathrm{d}\theta(X, Y) = X\langle Y, \theta\rangle - Y\langle X, \theta\rangle - \langle [X, Y], \theta\rangle$$

where θ is a one-form and X, Y vector fields.

Proposition 2.2. Suppose that H is a function of the form $H = (A+B)/(P+Q)$ with

$$\{A, B\} = \{P, Q\} = \{A, Q\} = \{P, B\} = 0.$$

Then $f = (AQ - BP)/(P+Q)$ is a constant of motion. This is easily proven by using the derivation properties of the Poisson bracket.

Proposition 2.3. Suppose that $H = \frac{1}{2}\boldsymbol{p}^2 + e(\boldsymbol{x}) + f(x)$ where f is an arbitrary function and e is homogeneous of degree minus two, i.e. satisfies $\boldsymbol{x}\cdot\nabla(e) + 2e = 0$. Then E is a constant of motion for H where

$$E = \boldsymbol{x}^2\boldsymbol{p}^2 - (\boldsymbol{x}\cdot\boldsymbol{p})^2 + 2\boldsymbol{x}^2 e.$$

Again the proof is a straightforward calculation.

3. Darboux's problem

Although it has been discussed several times before, I do not think that the problem posed earlier has ever been completely resolved (Darboux 1901, Whittaker 1937, Makarov *et al* 1967). We have

$$H = \tfrac{1}{2}\delta_{ij}p_i p_j + V(x), \qquad 1 \leqslant i, j \leqslant 2.$$

For a quadratic constant of motion f, it suffices to take f in the form $f = A_{ij}p_ip_j + A$ where A_{ij}, A are functions of the x's only (it is not difficult to see that a linear term in the p's itself must separately commute with H so its inclusion adds nothing new). The condition $\{H, f\} = 0$ gives two equations the first of which is

$$A_{(ij,k)} = 0. \tag{3.1a}$$

This defines A as a Killing tensor of the metric: in two dimensions these tensors form a six-dimensional vector space of quadratic polynomials in x and y. The second condition is

$$A_{,i} = 2V_{,j}A_{ij}. \tag{3.1b}$$

Integrability conditions on A yield

$$A_{ij}V_{,jk} - A_{kj}V_{,ji} + V_{,j}(A_{ij,k} - A_{kj,i}) = 0. \tag{3.2}$$

Here, $x = x^1$, $y = x^2$ so (3.2), since A_{ij} is Killing, gives for constants a, b_1, b_2, c_1, c_2, c_3

$$(axy + b_1x + b_2y - c_3)(V_{xx} - V_{yy}) - (a(x^2 - y^2) + 2b_2x - 2b_1y - c_1 + c_2)V_{xy}$$
$$+3(ay + b_1)V_x - 3(ax + b_2)V_y = 0. \tag{3.3}$$

By performing canonical transformations which leave $p_x^2+p_y^2$ invariant, (3.3) may be reduced to four different cases which are

$$xy(V_{xx}-V_{yy})-(x^2-y^2-c_1+c_2)V_{xy}+3yV_x-3xV_y=0, \tag{3.4}$$

$$xy(V_{xx}-V_{yy})-(x^2-y^2)V_{xy}+3yV_x-3xV_y=0, \tag{3.5}$$

$$(x+y)(V_{xx}-V_{yy})-(x-y)V_{xy}+3V_x-3V_y=0, \tag{3.6}$$

$$c_3(V_{xx}-V_{yy})+(c_2-c_1)V_{xy}=0. \tag{3.7}$$

In (3.4) it is assumed that $c^2=c_1-c_2\neq 0$ and in (3.7) that not both c_3, c_2-c_1 are zero as the corresponding constant of motion in that case is merely the Hamiltonian itself.

Equation (3.5) may be solved directly to give

$$V=e(x,y;-2)+f(x^2+y^2)$$

where f is an arbitrary function and $e(x,y;-2)$ indicates a homogeneous function of degree minus two. Proposition (2.3) gives the corresponding integral as

$$(yp_x-xp_y)^2+2(x^2+y^2)e. \tag{3.8}$$

For (3.4) define the canonical transformation

$$x=uv/c, \qquad y=c^{-1}[(u^2-c^2)(c^2-v^2)]^{1/2},$$

$$p_x=\frac{cv[(u^2-c^2)/(c^2-v^2)]^{1/2}p_u+cu[(c^2-v^2)/(u^2-c^2)]^{1/2}p_v}{v^2[(u^2-c^2)/(c^2-v^2)]^{1/2}+u^2[(c^2-v^2)/(u^2-c^2)]^{1/2}},$$

$$p_y=\frac{cup_u-cvp_v}{u^2[(c^2-v^2)/(u^2-c^2)]^{1/2}+v^2[(u^2-c^2)/(c^2-v^2)]^{1/2}}.$$

In these coordinates the solution to (3.4) is

$$V=(f(u)-g(v))/(u^2-v^2) \qquad \text{for arbitrary functions } f \text{ and } g.$$

The Hamiltonian is

$$2H=(u^2-v^2)^{-1}[(u^2-c^2)p_u^2+2f(u)]+(u^2-v^2)^{-1}[(c^2-v^2)p_v^2-2g(v)]. \tag{3.9}$$

From proposition (2.2) the corresponding constant of motion is

$$\frac{v^2}{u^2-v^2}[(u^2-c^2)p_u^2+2f(u)]+\frac{u^2}{u^2-v^2}[(c^2-v^2)p_v^2-2g(v)]. \tag{3.10}$$

Likewise for (3.6) define the canonical transformation

$$u=[2(x^2+y^2)]^{1/2}+x+y, \qquad v=[2(x^2+y^2)]^{1/2}-(x+y),$$

$$p_u=\frac{\{[2(x^2+y^2)]^{1/2}-2y\}p_x-\{[2(x^2+y^2)]^{1/2}-2x\}p_y}{4(x-y)},$$

$$p_v=\frac{\{[2(x^2+y^2)]^{1/2}+2y\}p_x+\{[2(x^2+y^2)]^{1/2}+2x\}p_y}{4(x-y)}.$$

The solution to (3.6) is

$$V=(g(u)+h(v))/(u+v) \qquad \text{for arbitrary functions } g, h.$$

The Hamiltonian may be written as

$$H=(u+v)^{-1}(4up_u^2+g(u))+(u+v)^{-1}(4vp_v^2+h(v)) \tag{3.11}$$

and again by proposition (2.2) the corresponding constant of motion is

$$-\frac{v}{u+v}(4up_u^2+g(u))+\frac{u}{u+v}(4vp_v^2+h(v)).$$

It remains only to discuss (3.8). There are a variety of cases depending on the values of c_1, c_2, c_3. However, in each case there is a canonical coordinate system (x', y', p'_x, p'_y) so that the Hamiltonian may be written as

$$H=\tfrac{1}{2}p_x'^2+V_1(x')+\tfrac{1}{2}p_y'^2+V_2(y') \tag{3.12}$$

for some functions V_1, V_2. Thus the Hamiltonian is additively separable and the constants of motion are obvious. It is interesting to observe that proposition (2.2) also applies to (3.12) and so we have established our claim that together proposition (2.2) and proposition (2.3) completely characterise all Hamiltonians with two degrees of freedom which admit quadratic integrals in addition to the Hamiltonian itself.

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